Pergamon

Int. J. Solids Structures Vol. 31, No. 19, pp. 2591–2597, 1994

0020–7683/94 $7.00 + .00

0020–7683(94)E0051–V

TWO-DIMENSIONAL ELASTOSTATIC GREEN'S FUNCTIONS FOR GENERAL ANISOTROPIC SOLIDS AND GENERALIZATION OF STROH'S FORMALISM

C.-Y. WANG
Center for Quality Engineering and Failure Prevention, Northwestern University, Evanston, IL 60208-3020, U.S.A.

(*Received* 24 *August* 1993; *in revised form* 21 *March* 1994)

Abstract—Explicit solutions for two-dimensional elastostatic Green's functions in general anisotropic solids are obtained by the use of an integral representation technique and a subsequent application of the residue calculus. These solutions lead naturally to the generalization of Stroh's formalism to include the most general class of anisotropic solids.

1. INTRODUCTION

To derive analytical solutions of two-dimensional elastostatic problems in anisotropic solids, a general solution known as Stroh's formalism is very elegant and productive (Stroh, 1962). The Green's function is one of the many solutions that can be deduced from Stroh's formalism [see, e.g. Barnett and Lothe (1974); Ting (1992); Hwu and Yen (1991)]. Stroh's formalism is given in terms of eigenvalues and eigenvectors. Hence it is implicit and, in its original form, limited to solids of which the eigenvalues are all distinct. The generalization of Stroh's formalism to include solids with non-distinct eigenvalues has been of interest to several researchers [see Nishioka and Lothe (1972a, b); Barnett and Lothe (1973); Lothe and Barnett (1976); Chadwick and Smith (1977); Ting (1982); Ting and Hwu (1988); Barnett (1992)]. In the last two references, all different cases of non-distinct eigenvalues have been considered. All the works mentioned above are, however, direct modifications of the original implicit form of Stroh's formalism, and are based on techniques to evaluate the limit as different eigenvalues approach a common limit.

In this paper, explicit expressions for the Green's function are obtained for the most general anisotropic solids. These expressions led naturally to the generalization of Stroh's formalism. The mathematical approach is based on the use of an integral representation technique and a subsequent application of the residue calculus. The results are obtained in terms of the residues of poles whose positions are given by the roots of the sextic equation of elasticity.

2. DEFINITION OF THE GREEN'S FUNCTION

Consider an unbounded homogeneous anisotropic linearly elastic solid subjected to a static line load uniformly distributed over the x_3-axis, in a fixed rectangular coordinate system, x_i. Thus, the response fields are independent of x_3. Denoted by $g_{pk}(x_1, x_2)$, the Green's function corresponds to the displacement field in the x_p-direction produced by the line load in the x_k-direction. Mathematically, the Green's function is defined as the solution of the following system of partial differential equations:

$$\Gamma_{ip}(\partial_1, \partial_2)g_{pk} = -\delta_{ik}\delta(\mathbf{x}) \tag{1}$$

where δ_{ik} is the Kronecker delta and

$$\Gamma_{ip}(\partial_1, \partial_2) = c_{i\alpha p\beta}\,\partial_\alpha\,\partial_\beta. \tag{2}$$

The elastic constants c_{ijpq} are fully symmetric and positive definite, i.e.

$$c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij} \tag{3}$$

and

$$c_{ijpq}e_{ij}e_{pq} > 0 \tag{4}$$

for any non-zero real symmetric tensor e_{ij}.

In this paper, a Roman suffix takes the values of 1, 2 and 3, while a Greek suffix takes the values of 1 and 2 only. The summation convention is applied over the range of the suffixes. In addition to suffix notations we also use bold-face letters for two-dimensional vectors, e.g. $\mathbf{x}$ has components x_α, and $\mathbf{s}\cdot\mathbf{x} = s_\alpha x_\alpha$ is the inner product. The derivative with respect to x_α is denoted by ∂_α.

3. ILLUSTRATION OF THE SOLUTION METHOD

For a simple exposition of the method of solution we will first consider the example of Laplace's equation. The method of solution will remain essentially the same for a general anisotropic solid.

Thus, let us consider a function $g(\mathbf{x})$ that satisfies

$$\Delta g(\mathbf{x}) = -\delta(\mathbf{x}). \tag{5}$$

The starting point of the derivation is the use of the following plane integral representation for $\delta(\mathbf{x})$:

$$\frac{1}{4\pi^2}\Delta\int_\Omega \frac{1}{|\mathbf{s}|^2}\log|\mathbf{s}\cdot\mathbf{x}|\omega(\mathbf{s}) = \delta(\mathbf{x}) \tag{6}$$

where Ω is any closed curve enclosing the origin point $\mathbf{s} = 0$ in $\mathbf{s}$ space, and

$$\omega(\mathbf{s}) = s_1\,\mathrm{d}s_2 - s_2\,\mathrm{d}s_1. \tag{7}$$

The proof of eqn (6) and details about the plane integral representation for arbitrary functions can be found in texts on the Radon transform [see, e.g. John (1955); Gel'fand *et al.* (1966); Wang and Achenbach (1994)].

It follows from eqn (6) that eqn (5) is satisfied by

$$g(\mathbf{x}) = \frac{-1}{4\pi^2}\int_\Omega \frac{1}{|\mathbf{s}|^2}\log|\mathbf{s}\cdot\mathbf{x}|\omega(\mathbf{s}). \tag{8}$$

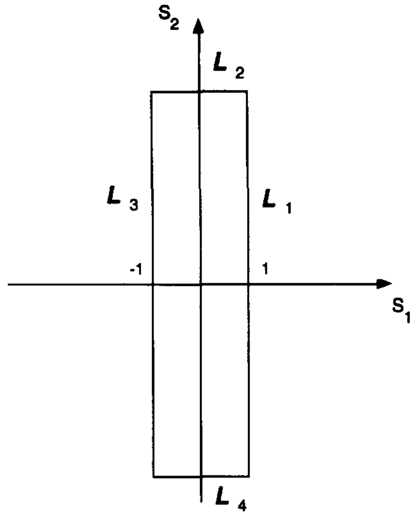

Fig. 1. Integral contour.

In order to apply the residue calculus to eqn (8), consider the closed contour given by

$$\Omega = L_1 + L_2 + L_3 + L_4 \tag{9}$$

shown in Fig. 1. It is easy to show that the contributions from L_2 and L_4 are zero as $|L_1| = |L_3| \to \infty$, and the contribution from L_3 equals the contribution from L_1. Thus eqn (8) reduces to

$$g(\mathbf{x}) = \frac{-1}{2\pi^2} \int_{L_1} \frac{1}{|\mathbf{s}|^2} \log |\mathbf{s} \cdot \mathbf{x}| \omega(\mathbf{s}) \tag{10}$$

$$= \frac{-1}{2\pi^2} \int_{-\infty}^{\infty} \frac{1}{1+s_2^2} \log |x_1 + s_2 x_2| \, \mathrm{d}s_2 \tag{11}$$

$$= \frac{-1}{2\pi^2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} \log (x_1 + \eta x_2) \, \mathrm{d}\eta. \tag{12}$$

Evaluating eqn (12) by the residue calculus yields the residue of the pole at $\eta = \mathrm{i}$ (or $-\mathrm{i}$). The result is

$$g(\mathbf{x}) = \frac{-1}{2\pi} \operatorname{Re} \log (x_1 + \mathrm{i} x_2) \tag{13}$$

$$= \frac{1}{2\pi} \log \left(\frac{1}{r} \right), \qquad \left(r = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2} \right). \tag{14}$$

Equation (14) is the well-known expression for the Green's function of Laplace's equation. Equation (13) is a special case of the well-known general solution:

$$f(z), \quad z = x_1 + \mathrm{i} x_2. \tag{15}$$

4. GENERAL SOLUTION OF THE GREEN'S FUNCTION

We now return to the Green's function defined by eqn (1). We observe that

$$\partial_\alpha f(\mathbf{s} \cdot \mathbf{x}) = s_\alpha \dot{f}(\mathbf{s} \cdot \mathbf{x}) \tag{16}$$

where an overdot denotes the differentiation with respect to the argument. By virtue of eqn (16), it can be shown that

$$\Gamma_{ip}(\partial_1, \partial_2) \int_\Omega \Gamma_{pk}^{-1}(\mathbf{s}) \log |\mathbf{s}\cdot\mathbf{x}|\omega(\mathbf{s}) = \delta_{ik}\Delta \int_\Omega \frac{1}{|\mathbf{s}|^2} \log|\mathbf{s}\cdot\mathbf{x}|\omega(\mathbf{s}) \tag{17}$$

where $\Gamma_{pk}^{-1}(\mathbf{s})$ is the inverse matrix of $\Gamma_{ip}(\mathbf{s})$ defined by

$$\Gamma_{ip}(\mathbf{s}) = \Gamma_{ip}(s_1, s_2) = c_{i\alpha p\beta} s_\alpha s_\beta. \tag{18}$$

It can be shown by virtue of eqns (3) and (4) that $\Gamma_{ip}(\mathbf{s})$ is symmetric and positive definite. Therefore $\Gamma_{pk}^{-1}(\mathbf{s})$ is well-defined. It follows from eqn (18) that $\Gamma_{ip}(\mathbf{s})$ is homogeneous of order 2, i.e. $\Gamma_{ip}(a\mathbf{s}) = a^2\Gamma_{ip}(\mathbf{s})$. Hence $\Gamma_{pk}^{-1}(\mathbf{s})$ is homogeneous of order -2.

It follows from eqns (6) and (17) that eqn (1) is satisfied by

$$g_{pk}(\mathbf{x}) = \frac{-1}{4\pi^2} \int_\Omega \Gamma_{pk}^{-1}(\mathbf{s}) \log|\mathbf{s}\cdot\mathbf{x}|\omega(\mathbf{s}). \tag{19}$$

Let the integral contour Ω be the same as in Section 3, indicated by eqn (9) and Fig. 1. By virtue of the homogeneity of $\Gamma_{pk}^{-1}(\mathbf{s})$ it can be easily shown that the contributions from L_2 and L_4 are zero as $|L_1| = |L_3| \to \infty$, and the contribution from L_3 equals that from L_1. Thus, similar to the derivation carried out in going from eqn (10) to eqn (12), we reduce eqn (19) to

$$g_{pk}(\mathbf{x}) = \frac{-1}{2\pi^2} \operatorname{Re} \int_{-\infty}^{\infty} \Gamma_{pk}^{-1}(1,\eta) \log(x_1 + \eta x_2)\, \mathrm{d}\eta \tag{20}$$

$$= \frac{-1}{2\pi^2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{A_{pk}(\eta)}{D(\eta)} \log(x_1 + \eta x_2)\, \mathrm{d}\eta \tag{21}$$

where

$$A_{pk}(\eta) = \operatorname{adj}[\Gamma_{pk}(1,\eta)] \tag{22}$$

$$D(\eta) = \det[\Gamma_{pk}(1,\eta)]. \tag{23}$$

We can see that $D(\eta)$ and $A_{pk}(\eta)$ are polynomial functions of η of order six and four, respectively. Since $\Gamma_{ip}(\mathbf{s})$ is positive definite, $D(\eta)$ does not have real roots. We also know that a polynomial of order N with real coefficients has N roots, and if $a+ib$ is a root, $a-ib$ must also be a root. Consequently, there are three roots satisfying

$$D(\eta_m) = 0 \tag{24}$$

with

$$\operatorname{Im}(\eta_m) > 0 \qquad (m = 1, 2, 3) \tag{25}$$

and we may write

$$D(\eta) = \sum_{k=0}^{6} a_k \eta^k = a_6 \prod_{m=1}^{3} (\eta - \eta_m)(\eta - \bar{\eta}_m) \tag{26}$$

where $\bar{\eta}_m$ are the conjugates of η_m and a_k are the coefficients of the sextic polynomial function $D(\eta)$. Equation (24) is called the sextic equation of elasticity (Head, 1979).

Applying the residue calculus to eqn (21) yields the summation of residues of poles at $\eta = \eta_m$ (or $\eta = \bar{\eta}_m$). When all three η_m are distinct, the result is

$$g_{pk} = \frac{\mathrm{Im}}{\pi} \sum_{m=1}^{3} \left[\frac{A_{pk}(\eta_m)}{\partial_\eta D(\eta_m)} \log(z_m) \right] \tag{27}$$

where

$$z_m = x_1 + \eta_m x_2. \tag{28}$$

For certain solids, e.g. isotropic solids, poles η_m defined by eqns (24)–(25) are not all simple. Generally, M ($1 \leqslant M \leqslant 3$) of these three poles are distinct. Let P_m denote the multiplicity of each pole, i.e. $P_m = 1, 2$ and 3 for a simple, double and triple pole. Solutions may then be generally written as

$$g_{pk} = \frac{\mathrm{Im}}{\pi} \sum_{m=1}^{M} \partial_\eta^{(P_m-1)} \left[\frac{A_{pk}(\eta_m)}{D_m(\eta_m)} \log(z_m) \right] \tag{29}$$

where

$$D_m(\eta) = \kappa_m D(\eta)/(\eta - \eta_m)^{P_m}. \tag{30}$$

In eqn (30) $\kappa_m = 1$ when $P_m = 1$ and 2, $\kappa_m = 0.5$ when $P_m = 3$. By virtue of eqn (26), we find that $D_m(\eta_m)$ is non-zero and finite. Clearly, eqn (27) is a special case of eqn (29) when $P_m = 1$ for all the poles.

Note that in eqns (27) and (29), and in the sequel the following notation is used:

$$\partial_\eta f(\eta_m) = [\partial f(\eta)/\partial \eta]_{(\eta = \eta_m)}. \tag{31}$$

5. GENERAL FORMALISM

It is interesting to note that once the Green's function in the form of eqn (29) is known, a general solution of similar form can be easily derived. To show this, let us consider the integral representation for a displacement field:

$$u_p(\mathbf{x}) = \int_S g_{pk}(\mathbf{x} - \mathbf{y}) \phi_k(\mathbf{y}) \, d\mathbf{y} \tag{32}$$

where $\phi_k(\mathbf{y})$ are arbitrary functions with a compact support S in the two-dimensional space R^2. It is known that eqn (32) yields the general form for a displacement field which satisfies

$$\Gamma_{ip}(\partial_1, \partial_2) \quad u_p(\mathbf{x}) = 0, \qquad \mathbf{x} \in \bar{S} \quad (S \cup \bar{S} = R^2). \tag{33}$$

Now, substitute eqn (29) into eqn (32) and set

$$f_k(z_m) = \frac{1}{\pi} \int_S \log(z_m - y_1 - \eta_m y_2) \phi_k(\mathbf{y}) \, d\mathbf{y}. \tag{34}$$

We obtain

$$u_p(\mathbf{x}) = \mathrm{Im} \sum_{m=1}^{M} \partial_\eta^{(P_m-1)} \left[\frac{A_{pk}(\eta_m)}{D_m(\eta_m)} f_k(z_m) \right]. \tag{35}$$

It can be shown that the complex form:

$$u_p(\mathbf{x}) = \sum_{m=1}^{M} \partial_\eta^{(P_m-1)} \left[\frac{A_{pk}(\eta_m)}{D_m(\eta_m)} f_k(z_m) \right] \tag{36}$$

also satisfies eqn (33). In this respect, eqn (36) defines the general solution. When $P_m = 1$ for all η_m, eqn (36) is the explicit form for the conventional Stroh's formalism.

6. DETAILS AND DISCUSSIONS

In this section we give some details of eqn (29) in the case of multiple poles. Since the case of a double pole is similar to the case of a triple pole, only the latter is discussed.

For a triple pole η_1, $M = 1$ and $P_1 = 3$, and eqn (29) yields

$$g_{pk} = \frac{\mathrm{Im}}{\pi} \left[\partial_\eta^2 \Omega_{pk}(\eta_1) \log(z_1) + 2\partial_\eta \Omega_{pk}(\eta_1) \frac{x_2}{z_1} - \Omega_{pk}(\eta_1) \frac{x_2^2}{z_1^2} \right] \tag{37}$$

where

$$\Omega_{pk}(\eta_1) = \frac{A_{pk}(\eta_1)}{D_1(\eta_1)}. \tag{38}$$

According to eqns (26) and (30), we have

$$D_1(\eta) = 0.5a_6(\eta - \bar{\eta}_1)^3. \tag{39}$$

We note that eqn (37) is quite different from eqn (27) (the case of simple poles), unless

$$\Omega_{pk}(\eta_1) = 0 \quad \text{and} \quad \partial_\eta \Omega_{pk}(\eta_1) = 0. \tag{40}$$

When the sextic equation of elasticity, eqn (24), has multiple roots, we say that (for the direction x_3) the solid is degenerate. In the modified implicit Stroh's eigenvalue formalism (Ting and Hwu, 1988; Barnett, 1992), degenerate solids are further separated into semisimple and non-semisimple solids. For semisimple solids, the conventional Stroh's formalism need not be modified. It can be shown that when eqn (40) holds the solids are semisimple. Otherwise, they are non-semisimple.

It is worth noting that isotropic solids belong to the class of materials for which

$$\Omega_{pk}(\eta_1) = 0 \quad \text{but} \quad \partial_\eta \Omega_{pk}(\eta_1) \neq 0. \tag{41}$$

Here we raise the question: does $\Omega_{pk}(\eta_1) = 0$ always holds when η_1 is a triple root of eqn (24)? So far no contradictions have been found in the literature (e.g. Ting, 1982), but the issue is yet to be resolved mathematically.

Acknowledgements—Helpful comments by Professor J. D. Achenbach are gratefully acknowledged. This work was carried out in the course of research sponsored by the Office of Naval Research under Grant N00014-89-J-1362.

REFERENCES

Barnett, D. M. (1992). On the non-existence of subsonic boundary-polarized two-component free surface waves. *Physica Scripta* **T44**, 98–103.

Barnett, D. M. and Lothe, J. (1973). Synthesis of the sextic and the integral formalism for dislocation, Green's functions and surface waves in anisotropic elastic solids. *Phys. Norv.* **7**, 13–19.

Barnett, D. M. and Lothe, J. (1974). An image force theorem for dislocations in anisotropic bicrystals. *J. Phys. F* **4**, 1618–1635.

Chadwick, P. and Smith, G. D. (1977). Foundations of the theory of surface waves in anisotropic elastic materials. *Adv. Appl. Mech.* **17**, 303–376.

Gel'fand, I. M., Graev, M. I. and Vilenkin, N. Ya. (1966). *Generalized Functions*, Vol. 5. Academic Press, New York.

Head, A. K. (1979). The Galois unsolvability of the sextic equation of anisotropic elasticity. *J. Elasticity* **9**, 9–20.

Hwu, C. and Yen, W. J. (1991). Green's functions of two-dimensional anisotropic plates containing an elliptic hole. *Int. J. Solids Structures* **27**, 1705–1719.

John, F. (1955). *Plane Waves and Spherical Means Applied to Partial Differential Equations*. Interscience, New York.

Lothe, J. and Barnett, D. M. (1976). On the existence of surface-wave solution for anisotropic elastic half-space with free surface. *J. Appl. Phys.* **47**, 428–433.

Nishioka, K. and Lothe, J. (1972a). Isotropic limiting behavior of the six-dimensional formalism of anisotropic dislocation theory and anisotropic Green's function theory, (I) sum rules and their applications. *Phys. Stat. Sol.* **B51**, 645–656.

Nishioka, K. and Lothe, J. (1972b). Isotropic limiting behavior of the six-dimensional formalism of anisotropic dislocation theory and anisotropic Green's function theory, (II) perturbation theory on the isotropic N-matrix. *Phys. Stat. Sol.* **B52**, 45–54.

Stroh, A. N. (1962). Steady state problems in anisotropic elasticity. *J. Math. Phys.* (Cambridge, MA) **41**, 77–103.

Ting, T. C. T. (1982). Effects of change of reference coordinates on the stress analyses of anisotropic elastic materials. *Int. J. Solids Structures* **18**, 139–152.

Ting, T. C. T. (1992). Image singularities of Green's functions for anisotropic elastic half spaces and bimaterials. *Q. J. Mech. Appl. Math.* **45**, 120–139.

Ting, T. C. T. and Hwu, C. (1988). Sextic formalism in anisotropic elasticity for almost non-semisimple matrix N. *Int. J. Solids Structures* **24**(1), 65–76.

Wang, C.-Y. and Achenbach, J. D. (1992). A new look at 2-D time-domain elastodynamic Green's functions for general anisotropic solids. *Wave Motion* **16**, 389–405.

Wang, C.-Y. and Achenbach, J. D. (1994). Elastodynamic fundamental solutions for anisotropic solids. *Geophysical J. Int.*, in press.